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Anomalous diffusion in correlated continuous time random walks

Vincent Tejedor and Ralf Metzler

Physics Department T30 g, Technical University of Munich, 85747 Garching, Germany

E-mail: metz@ph.tum.de

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Abstract
We demonstrate that continuous time random walks in which successive waiting times are correlated by Gaussian statistics lead to anomalous diffusion with the mean squared displacement $\langle \mathbf{r}^2(t)\rangle \simeq t^{2/3}$. Long-ranged correlations of the waiting times with a power-law exponent α ($0 < \alpha \leqslant 2$) give rise to subdiffusion of the form $\langle \mathbf{r}^2(t)\rangle \simeq t^{\alpha/(1+\alpha)}$. In contrast, correlations in the jump lengths are shown to produce superdiffusion. We show that in both cases weak ergodicity breaking occurs. Our results are in excellent agreement with simulations.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The continuous time random walk (CTRW) theory was introduced more than 40 years ago [1] to extend regular random walks on lattices to a continuous time variable. We characterize each jump in a CTRW process by a waiting time τ elapsing after the previous jump, and a variable jump length ξ. Each τ and ξ are independent random variables, identically distributed according to the probability densities $\psi(\tau)$ and $\lambda(\xi)$. For a power-law form $\psi(\tau) \simeq \tau^{-1-\beta}$ with $0 < \beta < 1$ the characteristic waiting time $\int_0^\infty \tau\psi(\tau)\,\mathrm{d}\tau$ diverges and the resulting CTRW is subdiffusive, $\langle \mathbf{r}^2(t)\rangle \simeq t^\beta$. For $\lambda(\xi) \simeq |\xi|^{-1-\gamma}$ with $0 < \gamma < 2$, the jump length variance $\int_{-\infty}^\infty \xi^2\lambda(\xi)\,\mathrm{d}\xi$ diverges and we obtain a superdiffusive Lévy flight. Spatiotemporal coupling of jump length and waiting time leads to Lévy walks with finite $\langle \mathbf{r}^2(t)\rangle \simeq t^\beta$ with $1 < \beta < 2$ [2]. The CTRW model was championed in the seminal work on charge carrier transport in amorphous semiconductors [3]. The CTRW theory has also been successfully applied in subsurface tracer dispersion [4], tick-tick dynamics in financial markets [5], cardiological rhythms [6], electron transfer [7], noise in plasma devices [8], dispersion in turbulent systems [9], search models [10] or in models of gene regulation [11], among many others [12]. CTRW models are closely related to the fractional Fokker–Planck equation [12, 13].

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In a CTRW process after each jump, a new pair of waiting time and jump lengths are drawn from the associated distributions, independent of the previous values. This independence of the waiting times and jump lengths giving rise to a renewal process is not always justified. As soon as the random walker has some form of memory, even a short one, the variables become non-independent. Examples are found in financial market dynamics [14], single trajectories in which there is a directional memory [15], or in astrophysics [16]. An important application is to search processes and human motion patterns in which memory and conscience will likely lead to a non-renewal situation. A general mathematical framework was developed for non-independent CTRWs [17]; however, it is quite cumbersome to apply and the cases solved so far only lead back to normal diffusion. An approach to coupling of waiting times based on the Langevin equation formulation of CTRW processes was recently introduced [18]. Some special cases were explored that bridge between CTRW and fractional Brownian motion [19]. Here we introduce a simple way to establish correlations in CTRW processes. Correlations between successive waiting times are shown to give rise to subdiffusion even when they are Gaussian, while correlations between jump lengths produce superdiffusion. We also consider long-ranged correlations and discuss anticorrelation effects. Our scaling arguments are in excellent accord with simulations.

After introducing the general framework we demonstrate how correlated waiting times lead to subdiffusion and weak ergodicity breaking. We then consider correlations in the jump length, and proceed to analyse anticorrelation effects. Finally, we provide some details on the derivations and draw our conclusions.

2. General framework

Let us briefly review the general framework for correlated CTRWs from [17] for correlated waiting times. Then the waiting time ψ_n for a given step n is conditioned by the previous waiting time ψ_{n-1}, as quantified by the joint probability $P(\psi_n, \psi_{n-1}) = P(\psi_n|\psi_{n-1})P(\psi_{n-1})$. Here $P(\psi_{n-1})$ is the probability of having the waiting time ψ_{n-1} in the previous step, and $P(\psi_n|\psi_{n-1})$ is the conditional probability of finding a waiting time ψ_n for given ψ_{n-1}. The normalization conditions are

$$\begin{aligned}&\int_0^\infty P(\psi_n)\,\mathrm{d}\psi_n = 1, \qquad \int_0^\infty\int_0^\infty P(\psi_n, \psi_{n-1})\,\mathrm{d}\psi_n\,\mathrm{d}\psi_{n-1} = 1,\\ &\int_0^\infty P(\psi_n|\psi_{n-1})\,\mathrm{d}\psi_n = 1.\end{aligned} \tag{1}$$

By recurrence, we obtain the joint probability

$$P(\psi_n, \psi_{n-1}, \ldots, \psi_0) = \prod_{k=1}^{n} P(\psi_k|\psi_{k-1})P(\psi_0), \tag{2}$$

demonstrating that the waiting time ψ_n in fact depends on all previous waiting times. Note that in the decoupled case $P(\psi_k|\psi_{k-1}) = f(\psi_k)g(\psi_{k-1})$, we get back to a regular renewal CTRW.

The marginal probability of ψ_n is defined as $P(\psi_n) = \int_0^\infty P(\psi_n|\psi_{n-1})P(\psi_{n-1})\,\mathrm{d}\psi_{n-1}$. According to equation (2), this leads to an n-fold integration over the product on its right-hand side. In the general case, it is quite hard to compute this quantity, and this is why in previous literature only normal diffusion was treated in this framework.

3. Random walk of waiting times

Instead of constructing the process from definitions (1) and (2) we start from a different angle. Namely we build the value of waiting time ψ_n from the previous waiting time plus a random deviation $\delta\psi_n$:

$$\psi_n = \psi_{n-1} + \delta\psi_n. \tag{3}$$

The sequence of waiting times can therefore be viewed as a random walk in the space of waiting times. Similarly, we will proceed with coupled jump lengths below. The waiting time ψ_n can therefore be expressed through $\psi_n = \sum_{i=0}^{n} \delta\psi_i$, where we assumed that $\psi_0 = \delta\psi_0$, without loss of generality. A reflecting boundary condition at $\psi = 0$ ensures that the waiting times are positive. If the random variations $\delta\psi_n$ are normally distributed, we can obtain the following conditional probability:

$$P(\psi_n|\psi_{n-1}) = \frac{1}{\sqrt{4\pi\sigma^2}}\left[\exp\left(-\frac{(\psi_n-\psi_{n-1})^2}{4\sigma^2}\right) + \exp\left(-\frac{(\psi_n+\psi_{n-1})^2}{4\sigma^2}\right)\right]. \tag{4}$$

The mean squared displacement (MSD) for this process grows with the number of steps as $\langle(\Delta\psi(n))^2\rangle \sim 2\sigma^2 n$ for large n. To proceed, we compute the probability $P(t, n)$ to have made n steps up to time t. In the Laplace space[1]

$$P(s,n) = \int_0^\infty P(t,n)\,\mathrm{e}^{-st}\mathrm{d}t = \frac{1-\psi_n(s)}{s}\prod_{i=0}^{n-1}\psi_i(s). \tag{5}$$

After some calculations we arrive at (see the appendix)

$$P(s,n) = \frac{1}{s}\left[\delta\psi\left(\sqrt{\frac{n(n+1)(2n+1)}{6}}s\right) - \delta\psi\left(\sqrt{\frac{(n+1)(n+2)(2n+3)}{6}}s\right)\right]. \tag{6}$$

We obtain the Laplace transform of the mean number of steps by summation, $\langle n(s)\rangle = \sum_{n=0}^{\infty} nP(s,n)$. With the approximations detailed in the appendix, we find in a leading order around $s = 0$ (corresponding to long times),

$$\langle n(s)\rangle \sim \frac{1}{s}\frac{3^{1/3}\Gamma(5/3)}{(s\tau)^{2/3}} \Rightarrow \langle n(t)\rangle \sim \left(\frac{t}{\sigma}\right)^{2/3}, \tag{7}$$

where we assumed $\tau \propto \sigma$. At long times the Gaussian waiting time correlation results in the subdiffusion law [20]

$$\langle \mathbf{r}^2(t)\rangle = \langle\delta\mathbf{r}^2\rangle\langle n(t)\rangle \sim Kt^{2/3}, \tag{8}$$

where $\langle\delta\mathbf{r}^2\rangle$ is the jump length variance of the process and $K = \langle\delta\mathbf{r}^2\rangle/\sigma^{2/3}$ the generalized diffusion constant. Equation (8) is one of the main results of this work. We stress again that this subdiffusion emerges from a random process that has a finite waiting time in each step. However, as time proceeds this waiting time slowly diverges $[\langle\Delta\psi(n)\rangle \sim n^{1/2}]$, due to the diffusive coupling of waiting times.

If we take a sharp correlation of waiting times with $\sigma = 0$, equation (4) leads to $P(\psi_n) = \delta(\psi_n - \psi_0) + \delta(\psi_n + \psi_0)$. At each step the waiting time is the same. The mean waiting time is $\langle\psi_n\rangle = \int_0^\infty \psi\,[\delta(\psi-\psi_0) + \delta(\psi+\psi_0)]\,\mathrm{d}\psi = \psi_0$ such that we find the classical Brownian motion with the diffusion coefficient $\langle\delta\mathbf{r}^2\rangle/\psi_0$, as it should be.

Consider now the case of a stable distribution with index α for $\delta\psi$. For $1 < \alpha \leqslant 2$ the first moment $\langle\delta\psi\rangle = \tau$ of the random walk in ψ space is still finite. We then follow similar steps as outlined above, obtaining

$$\langle n(t)\rangle \sim (t/\sigma)^{\alpha/(\alpha+1)} \Rightarrow \langle\mathbf{r}^2(t)\rangle \simeq Kt^{\alpha/(\alpha+1)}. \tag{9}$$

[1] We denote the Laplace transform by explicit dependence on the variable: $f(s) = \mathscr{L}\{f(t)\} = \int_0^\infty f(t)\exp(-st)\,\mathrm{d}t$.

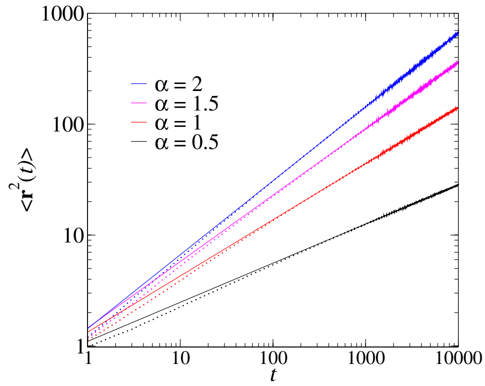

Figure 1. $\langle \mathbf{r}^2(t)\rangle$ for a waiting time correlated 3D Gaussian walk. The $\delta\psi$ follow an α-stable law with scale factor $c = 1$; α decreases from top to bottom. Simulations (—) and power-laws $(\cdots)$ with fitted exponents 0.35, 0.50, 0.60, 0.66. Theoretical values $\alpha/(\alpha+1)$: 0.33, 0.50, 0.60, 0.66.

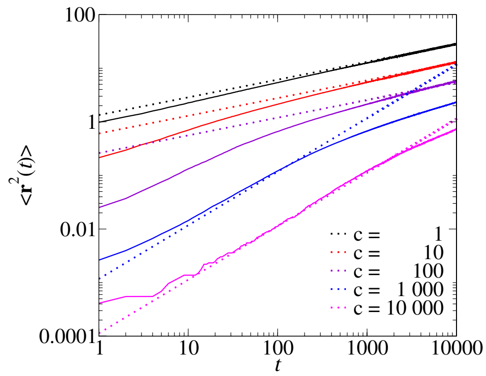

Figure 2. Same as figure 1 with $\alpha = 0.5$ and various scale factors c (c increases from top to bottom). Simulations (—) and power-laws $(\cdots)$ with slope 0.33 ($c = 1$, 10, 100) and 1 ($c = 1.000$, 10.000). Note the turnover to slope $\alpha/(\alpha+1)$ at $t \sim c$. Compare with the text.

The case $0 < \alpha \leqslant 1$ is somewhat more involved. We argue that for any stable distribution, we have $\sum_{i=1}^{n} \psi_i(t) \overset{d}{\sim} n^{-(\alpha+1)/\alpha}\delta\psi(n^{-(\alpha+1)/\alpha}t)$, where $\overset{d}{\sim}$ is a scaling equality of distributions. This leads to the relation $t(n)/\sigma \overset{d}{\sim} n^{(\alpha+1)/\alpha}$, such that equation (9) still holds for any $0 < \alpha \leqslant 2$. We note that result (9) was retrieved from a Langevin equation approach in [18].

For a stable distribution the jumps between successive waiting times become increasingly larger when α is decreased, affecting even slower diffusion $\langle \mathbf{r}^2(t)\rangle \simeq t^{\alpha/(\alpha+1)}$. In the limit $\alpha = 2$, we are back to the Gaussian diffusion in ψ space and a 2/3 subdiffusion in position space, $\langle \mathbf{r}^2(t)\rangle \simeq t^{2/3}$. In figure 1 we demonstrate excellent agreement between our analytical findings and simulations results of random processes performed with stable correlations between successive waiting times.

The characteristic function of the stable variable $\delta\psi$ is $\phi_{\delta\psi}(q) = \exp(-|cq|^\alpha)$, see [20]. While at long times the predicted subdiffusive behaviour is attained, we observe a transient regime of normal diffusion, $\langle \mathbf{r}^2(t)\rangle \simeq t$, when the scale factor c increases (see figure 2). At short times, we may neglect the probability that the random walker makes more than one step.

The probability of making the first step corresponds to the cumulative function of the waiting time distribution $F_{\delta\psi}(t)$,

$$F_{\delta\psi}(t) = \frac{2}{\pi}\int_0^\infty \frac{\sin(tq)}{q}\phi_{\delta\psi}(q)\,\mathrm{d}q \sim \frac{2\Gamma(1+1/\alpha)}{\pi c}t. \tag{10}$$

Thus, the initial linear slope in the ensemble average is due to the linearity of the cumulative function for short waiting times. This effect vanishes as soon as the probability increases that the walker makes two or more steps, and converges to the predicted long-time behaviour (9).

3.1. Weak ergodicity breaking

When dealing with the time series of a single particle trajectory of length T, instead of the ensemble averaged MSD $\langle\mathbf{r}^2(t)\rangle$ one calculates the time-averaged MSD

$$\overline{\delta^2(\Delta, T)} = \frac{1}{T-\Delta}\int_0^{T-\Delta}[\mathbf{r}(t+\Delta)-\mathbf{r}(t)]^2\,\mathrm{d}t, \tag{11}$$

relating two positions separated by the lag time Δ. Ensemble averaging equation (11) the square brackets become $\langle[\mathbf{r}(t+\Delta)-\mathbf{r}(t)]^2\rangle = \langle\delta\mathbf{r}^2\rangle\langle n_{t,t+\Delta}\rangle$ where $\langle\delta\mathbf{r}^2\rangle$ is the (finite) variance of jump lengths, and $\langle n_{t,t+\Delta}\rangle$ counts the average number of jumps in the time interval $[t, t+\Delta]$. For the normal diffusion $\langle n(t)\rangle = t/\tau$ and therefore $\overline{\delta^2(\Delta, T)} = 2\,\mathrm{d}K\Delta$ behaves exactly as the ensemble average such that the lag time Δ is exchangeable with the process time t of the ensemble average. In contrast, in a subdiffusive renewal CTRW with the waiting time density $\psi(t) \simeq t^{-1-\alpha}$ $(0 < \alpha < 1)$, while $\langle\mathbf{r}^2(t)\rangle \simeq t^\alpha$, it turns out that $\overline{\delta^2(\Delta, T)} \simeq \Delta/T^{1-\alpha}$ for $\Delta \ll T$, i.e. we observe a so-called weak ergodicity breaking [21, 22]. Do we observe similar behaviour for our coupled CTRW? With equation (7) and the relation $\langle n_{t,t+\Delta}\rangle = \langle n_{0,t+\Delta}\rangle - \langle n_{0,t}\rangle$,

$$\langle[\mathbf{r}(t+\Delta)-\mathbf{r}(t)]^2\rangle \sim 2K[(t+\Delta)^{\alpha/(1+\alpha)} - t^{\alpha/(1+\alpha)}], \tag{12}$$

with $K = \langle\delta\mathbf{r}^2\rangle/(2\,\mathrm{d}\sigma^{\alpha/(1+\alpha)})$, and therefore

$$\left\langle\overline{\delta^2(\Delta, T)}\right\rangle \sim 2\,\mathrm{d}K\Delta/T^{\alpha/(1+\alpha)}. \tag{13}$$

Thus, our non-renewal, coupled CTRW also exhibits a weak ergodicity breaking, even in the limit $\alpha = 2$ of Gaussian waiting time coupling. Simulations of this process indeed confirm the predicted scaling of the time averaged MSD with both lag time Δ and measurement time T. In figure 3 we demonstrate that individual trajectories show significant scatter in their amplitude, as observed for renewal CTRW subdiffusion [21].

4. Correlated jump lengths

Let us now consider a random walk with constant waiting time $\psi = 1$ but correlated jump lengths. If $\mathbf{r}(t)$ is the position of the walker we define $\mathbf{x}(t) = \mathbf{r}(t) - \mathbf{r}(t-1)$ $(\mathbf{x}(0) = 0)$. The jump length is now assumed to diffuse in $\mathbf{x}$ space, with increments $\delta\mathbf{x}(t) = \mathbf{x}(t) - \mathbf{x}(t-1)$ that are normally distributed with mean 0 and variance σ. We find

$$P(\mathbf{x}(t) = (x, y, z)) = \frac{1}{(\pi\sigma^2 t)^{3/2}}\exp\left(-\frac{x^2+y^2+z^2}{\sigma^2 t}\right), \tag{14}$$

with variance $\langle\mathbf{x}^2(t)\rangle = \frac{3}{2}\sigma^2 t$. For this process we have $\mathbf{r}(t) = \mathbf{r}(0) + \sum_{i=1}^t \mathbf{x}(i)$ and obtain (see the appendix)

$$\langle[\mathbf{r}(t)-\mathbf{r}(0)]^2\rangle = \frac{t(t+1)(2t+1)\sigma^2}{4}. \tag{15}$$

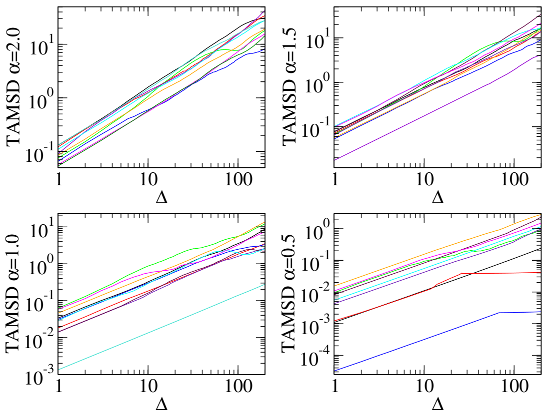

Figure 3. Time-averaged MSD as a function of Δ for a CTRW with correlated waiting time ($T = 2000$). The plots show a scatter between individual trajectories that becomes more pronounced for decreasing α. The data approximately scale as the ensemble average, equation (13). For very small α we observe a plateau due to the occurrence of extremely long waiting times of the order of T.

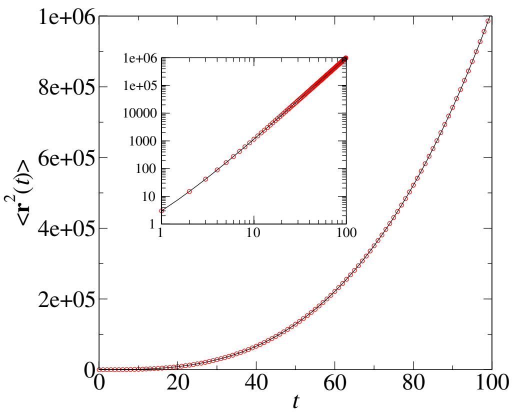

Figure 4. 3D random walk with correlated jump lengths. We chose $\mathbf{x}(0) = 0$. $\delta\mathbf{x}(t)$ is normally distributed with $\sigma = \sqrt{2}$. Simulations (○) and theory (—), equation (15).

Thus, when we assume a diffusion in the space of jump lengths we obtain a Richardson-type t^3 scaling. Figure 4 demonstrates excellent agreement of equation (15) with the simulations result. For a fixed jump length ($\sigma = 0$) all steps have the same length and direction, and the corresponding random walk is ballistic, $\langle\mathbf{r}^2(t)\rangle = \mathbf{x}^2(0)t^2$.

Classical diffusion cannot be retrieved with this mechanism, as this would require directional randomness for jumping left/right. In fact we obtain for the jump correlations that $\langle\mathbf{x}(t)\cdot\mathbf{x}(t+\tau)\rangle = \frac{3}{2}\sigma^2 t$. The position correlations become

$$\langle\mathbf{r}(t)\cdot\mathbf{r}(t+\tau)\rangle = \frac{t(t+1)(2t+1)\sigma^2}{4} + \frac{3\sigma^2 t(t+1)\tau}{4}. \qquad (16)$$

If the increments $\delta\mathbf{x}$ are distributed according to a stable distribution of exponent α and scale factor c, each coordinate of $\mathbf{r}(t)$ is distributed according to a stable distribution

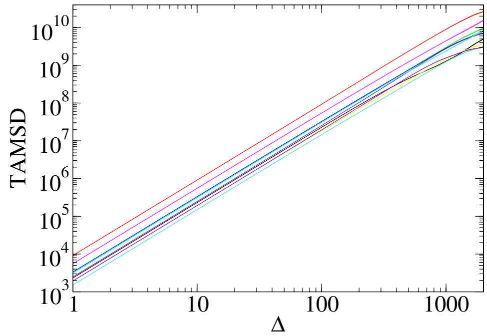

Figure 5. Time-averaged MSD as a function of Δ for ten different trajectories of length $T = 2000$ with the jump length correlation. We chose $\sigma^2 = 2$.

of exponent α and scale factor $c(t) = \left(\sum_{i=1}^{t} i^{\alpha}\right)^{1/\alpha} c \simeq ct^{(1+\alpha)/\alpha}$. The mean squared displacement diverges for $0 < \alpha < 2$ but we observe the superdiffusive scaling $x \sim t^{(1+\alpha)/\alpha}$. In the limit $\alpha = 2$, using $2c^2 = \sigma^2$ we retrieve the previous result (15), while for $\alpha \to 0$ we find the regular Lévy flight scaling $x \sim t^{1/\alpha}$ [12].

4.1. Weak ergodicity breaking

The process with coupled jump lengths has the same waiting time for each jump. Could it still be subject to weak ergodicity breaking? Combining equations (11) and (16) we obtain the equality

$$\left\langle \overline{\delta^2(\Delta, T)} \right\rangle = \frac{3\sigma^2}{4} \Delta^2 T + \sigma^2 \left(\frac{\Delta}{4} + \frac{3\Delta^2}{4} - \frac{\Delta^3}{4} \right). \tag{17}$$

In the limit $\Delta \ll T$ we find the scaling $\langle \overline{\delta^2}(\Delta, T) \rangle \simeq \Delta^2 T$, contrasting the leading cubic behaviour $\langle \mathbf{r}^2(t) \rangle \simeq t^3$ of the ensemble average. Thus, also the non-renewal CTRW with coupled jump lengths leads to a weak ergodicity breaking. Again, simulations corroborate the scaling with lag time Δ and overall process time T. In figure 5 we show the scatter between different trajectories.

5. Anticorrelated waiting times

Another way to introduce correlations is to consider diffusion on a network of nodes on each of which a different distribution is assumed. We illustrate this approach by anticorrelated waiting times. Namely we have two waiting time distributions, Ψ_1 and Ψ_2, such that Ψ_1 is centred around short waiting times and Ψ_2 around longer ones. We start by choosing a waiting time from one of the Ψ_i and then pass from one waiting time distribution to the other according to the following rules: (i) if we are in a state Ψ_1 and get a waiting time shorter than a preset t_1, we shift to Ψ_2 for the next step. Otherwise we remain with Ψ_1. (ii) If we are in a state Ψ_2 and find a waiting time longer than a given t_2, we shift to Ψ_1, otherwise stay in Ψ_2.

We can view this process as a diffusion in a network with two nodes. Being at node i at step n means that the nth waiting time is extracted from distribution Ψ_i. The transition probabilities between the two nodes are $P_1 = \int_0^{t_1} \Psi_1(t)\, \mathrm{d}t$ and $P_2 = \int_{t_2}^{\infty} \Psi_2(t)\, \mathrm{d}t$. The probability that the nth waiting time is chosen from the distribution Ψ_1 is $\Pr(\Psi_1, n) =$

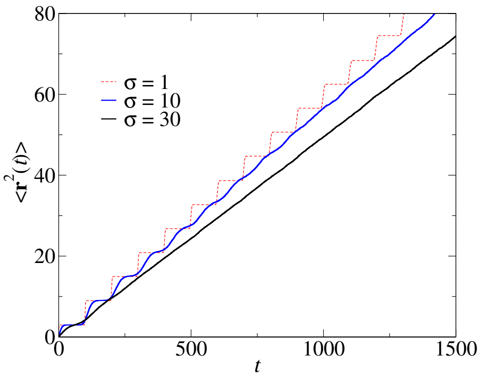

Figure 6. Anticorrelated CTRW for a 3D Gaussian walk. ψ_1 and ψ_2 are normal distributions centred around 0 and 100. The shift times are $t_1 = t_2 = 50$.

$(1-P_1)\mathrm{Pr}(\Psi_1, n-1)+P_2\mathrm{Pr}(\Psi_2, n-1)$, and similarly for Ψ_2. These two equations can easily be solved. With a given initial condition we find

$$\mathrm{Pr}(\Psi_1, n) = \frac{P_2 + \lambda^n(P_1\mathrm{Pr}(\Psi_1, 0) - P_2\mathrm{Pr}(\Psi_2, 0))}{P_1 + P_2}, \tag{18}$$

with $\lambda = 1 - P_1 - P_2$. For $n \to \infty$ we converge to the equilibrium probability $\mathrm{Pr}(\Psi_1) = P_2/(P_1 + P_2)$. If both distributions have a finite characteristic waiting time we see that at $n \to \infty$, $\langle\psi\rangle = (P_2\langle\psi_1\rangle + P_1\langle\psi_2\rangle)/(P_1 + P_2)$.

Consider two limits: (i) if $P_1 = 1$ and $P_2 = 1$ (i.e. $t_1 \to \infty$ and $t_2 = 0$) we change the waiting time at each step. Roughly, if we begin with the distribution Ψ_1 we expect $t = n(\langle\psi_1\rangle + \langle\psi_2\rangle)/2 + [1 - (-1)^n](\langle\psi_2\rangle - \langle\psi_1\rangle)/4$. If $\langle\psi_1\rangle \ll \langle\psi_2\rangle$, at the beginning we have a short waiting time alternating with a long one. After a long time $t \gg \langle\psi_2\rangle$, however, we see a smooth curve for the mean squared displacement, see figure 6. (ii) If $P_1 \ll 1$, we have a classical CTRW governed by distribution ψ_1 that after a while is somewhat modified by distribution ψ_2. At long times we find the same result as in case (i). This model can easily be extended to a wider network of nodes.

6. Conclusions

We demonstrated that the elusive coupled, non-renewal CTRW can indeed be applied to non-Brownian processes. Compared to previous models we believe that the idea of diffusion in the space of waiting times or jump length is quite intuitive and generic, such that this model will lend itself to a broad class of phenomena. In particular, this approach allows us to consider Lévy stable correlations in waiting times and jump lengths, significantly generalizing previous results. We find that correlated waiting times lead to subdiffusion while correlations in the jump lengths give rise to superdiffusion, even when the waiting time or jump length increments are Gaussian. We also showed that anticorrelations in the long-time limit produce normal diffusion.

Both temporal and spatial correlations are demonstrated to lead to weak ergodicity breaking: the long-time average of the mean squared displacement of a single trajectory shows different scaling behaviour than the corresponding ensemble average. Surprisingly this is also the case for jump length correlations in which successive jumps are separated by constant waiting times.

Acknowledgments

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Appendix. Explicit derivation of the jump statistics

We here show how the average number of jumps for correlated waiting time, and the position correlations are calculated.

A.1. Waiting time correlation

To pass from equation (5) to (6) we note that

$$\sum_{i=0}^{n-1} \psi_i = \sum_{i=0}^{n-1} \sum_{j=0}^{i} \delta\psi_j = \sum_{i=0}^{n-1} (n-i)\delta\psi_i. \tag{A.1}$$

For a stable law we know that the sum Y of independent, identically distributed random variables X fulfils [20]

$$Y = \sum_{k=1}^{n} a_k X_k \Rightarrow Y(s) = X\left(\left(\sum_{k=1}^{n} a_k^\alpha\right)^{1/\alpha} s\right). \tag{A.2}$$

With $\sum_{i=0}^{n-1}(n-i)^2 = \sum_{i=1}^{n} i^2 = n(n+1)(n+2)/6$ we see that in the Laplace domain

$$\prod_{k=0}^{n-1} \psi_k(s) = \delta\psi\left(\sqrt{\frac{n(n+1)(2n+1)}{6}}s\right), \tag{A.3}$$

leading to equation (6). The mean number of steps, $\langle n(s)\rangle = \sum_{n=0}^{\infty} P(s,n)$is then found to be

$$\langle n(s)\rangle = \frac{1}{s}\sum_{n=1}^{\infty} \delta\psi\left(\sqrt{\frac{n(n+1)(2n+1)}{6}}s\right). \tag{A.4}$$

Due to its Gaussian nature we know that $\delta\psi(s) = 1 - \tau s + o(s)$, where τ is the mean waiting time for a Gaussian random variable of variance σ. As we are only interested in the behaviour at long times, i.e. small s, we approximate $\delta\psi(s) \sim \exp(-\tau s)$ and derive the leading contribution of $\langle n(s)\rangle$ around $s = 0$. With $n(n+1)(2n+1)/6 \approx n^3/3$,

$$\langle n(s)\rangle \sim \frac{1}{s}\sum_{n=1}^{\infty} \exp(-[n^3/3]^{1/2} s\tau). \tag{A.5}$$

Approximating the sum by an integral, we finally obtain equation (7). For a Lévy stable statistics of the increments ψ_i, we use the characteristic function $\delta\psi(\omega) = \exp(-c|\omega|^\alpha)$ in Fourier space denoted by the frequency ω. We find

$$\prod_{i=0}^{n-1} \psi_i(\omega) = \exp\left(-\sum_{i=1}^{n} i^\alpha |c\omega|^\alpha\right), \tag{A.6}$$

from which we obtain the average number of steps

$$\langle n(\omega)\rangle = \sum_{n=0}^{\infty} n\frac{1 - \exp(-(n+1)^\alpha |c\omega|^\alpha)}{i\omega}\, \mathrm{e}^{-\sum_{i=1}^{n} i^\alpha |c\omega|^\alpha}. \tag{A.7}$$

After approximation of the harmonic number $\sum_{i=1}^{n} i^{\alpha} \approx n^{1+\alpha}/(1+\alpha)$ and turning from sum to integral, we find after Fourier transform

$$\langle n(t)\rangle \sim \frac{(\alpha+1)^{1/(\alpha+1)}}{2\cos\left(\alpha\pi/[2(\alpha+1)]\right)}\left(\frac{t}{c}\right)^{\alpha/(\alpha+1)}. \tag{A.8}$$

In the limit $\alpha = 2$ we return to expression (7).

A.2. Jump length correlation

Assume the Gaussian distribution (14) for the jump displacement $\mathbf{x}(t) = \mathbf{r}(t) - \mathbf{r}(t-1)$ in the tth jump, with initial condition $\mathbf{x}(0) = 0$. The incremental change of the jump lengths is $\delta\mathbf{x}(t) = \mathbf{x}(t) - \mathbf{x}(t-1)$. Consequently, $\langle \mathbf{x}^2(t)\rangle = \frac{3}{2}\sigma^2 t$. We can then calculate the MSD ($\Delta\mathbf{r}(t) = \mathbf{r}(t) - \mathbf{r}(0)$),

$$\begin{aligned}
\langle[\Delta\mathbf{r}(t)]^2\rangle &= \left\langle\left(\sum_{i=1}^{t}\mathbf{x}(i)\right)^2\right\rangle = \left\langle\left(\sum_{i=1}^{t}\sum_{j=1}^{i}\delta\mathbf{x}(j)\right)^2\right\rangle \\
&= \left\langle\left(\sum_{i=1}^{t}(t+1-i)\delta\mathbf{x}(i)\right)^2\right\rangle \\
&= \sum_{i=1}^{t}\sum_{j=1}^{t}(t+1-i)(t+1-j)\langle\delta\mathbf{x}(i)\cdot\delta\mathbf{x}(j)\rangle.
\end{aligned} \tag{A.9}$$

Due to independence of the increments, $\langle\delta\mathbf{x}(i)\cdot\delta\mathbf{x}(j)\rangle = \frac{3}{2}\sigma^2\delta_{ij}$, we obtain the exact relation (15).

For the jump correlation $\langle\mathbf{r}(t)\cdot\mathbf{r}(t+\tau)\rangle$, we start with

$$\langle\mathbf{x}(t)\cdot\mathbf{x}(t+\tau)\rangle = \left\langle\mathbf{x}(t)\cdot\left(\mathbf{x}(t)+\sum_{i=1}^{\tau}\delta\mathbf{x}(t+i)\right)\right\rangle. \tag{A.10}$$

As $\mathbf{x}(t)$ and $\delta\mathbf{x}(t+i)$ are uncorrelated and of mean 0, we obtain $\langle\mathbf{x}(t)\cdot\mathbf{x}(t+\tau)\rangle = \frac{3}{2}\sigma^2 t$. This expression allows us to obtain the position correlation

$$\langle\mathbf{r}(t)\cdot\mathbf{r}(t+\tau)\rangle = \left\langle\mathbf{r}(t)\cdot\left(\mathbf{r}(t)+\sum_{i=1}^{\tau}\mathbf{x}(t+i)\right)\right\rangle. \tag{A.11}$$

Since $\mathbf{x}(t)$ have zero mean we obtain

$$\langle\mathbf{r}(t)\cdot\mathbf{r}(t+\tau)\rangle = \langle(\mathbf{r}(t))^2\rangle + \sum_{i=1}^{t}\sum_{j=1}^{\tau}\langle\mathbf{x}(i)\cdot\mathbf{x}(t+j)\rangle. \tag{A.12}$$

As $\langle\mathbf{x}(i)\cdot\mathbf{x}(t+j)\rangle = \frac{3}{2}\sigma^2 i$, we arrive at equation (16).

For a Lévy distribution of the jump increments the position distribution of the overall process is stable with index α and scale factor $c(t) \sim ct^{(1+\alpha)/\alpha}/(1+\alpha)^{1/\alpha}$. If we concentrate on the x coordinate, the characteristic function becomes $P(q,t) = \exp(-|qc(t)|^{\alpha})$. While the variance of this law diverges, one can calculate fractional moments of order $0 < \delta < \alpha$ [12], scaling like $\langle|x|^{\delta}(t)\rangle \simeq c(t)^{\delta}$. We therefore find the superdiffusive scaling $x^2 \sim t^{2(1+\alpha)/\alpha}$. In the limit $\alpha \to 2$, this reproduces the cubic scaling $x^2 \sim t^3$ found for Gaussian correlations in the jump increments. For decreasing α the superdiffusion is enhanced, and in the limit of small α we find the scaling $x^2 \sim t^{2/\alpha}$ of a regular Lévy flight, i.e. a renewal process with a Lévy stable jump length distribution $\lambda(\xi)$ of index α.

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